

BALANCED SUBDIVISIONS AND FLIPS ON SURFACES

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ABSTRACT. In this paper, we show that two balanced triangulations of a closed surface are not necessarily connected by a sequence of balanced stellar subdivisions and welds. This answers a question posed by Izvestiev, Klee and Novik. We also show that two balanced triangulations of a closed surface are connected by a sequence of three local operations, which we call the pentagon contraction, the balanced edge subdivision and the balanced edge weld. In addition, we prove that two balanced triangulations of the 2-sphere are connected by a sequence of pentagon contractions and their inverses if none of them are octahedral spheres.

1. INTRODUCTION

It is a classical result in the combinatorial topology [Al] which shows that two PL-homeomorphic simplicial complexes are connected by a sequence of stellar subdivisions and their inverses. A closely related result is Pachner's result [Pa1, Pa2] which shows that two PL-homeomorphic combinatorial manifolds are connected by a sequence of bistellar flips (see also [Li] for the proofs of both results). A combinatorial d -manifold is a triangulation of a d -manifold all whose vertex links are PL $(d-1)$ -spheres. A combinatorial d -manifold is said to be *balanced* if its graph is $(d+1)$ -colorable. Recently, Izvestiev, Klee and Novik [IKN] proved an analogue of Pachner's result for balanced combinatorial manifolds. They introduced a version of bistellar flips that preserves the balanced property, which they call *cross-flips*, and proved that two PL-homeomorphic balanced combinatorial manifolds are connected by a sequence of cross-flips. In this paper, we study the following questions related to their result in the special case of triangulated surfaces.

- There is an analogue of stellar subdivisions for balanced simplicial complexes, called *balanced stellar subdivisions* (see [IKN, §2.5]). Are two PL-homeomorphic combinatorial manifolds connected by a sequence of balanced stellar subdivisions and their inverses?
- It is known that not all cross-flips are necessary to connect any two PL-homeomorphic balanced combinatorial manifolds. How many different types of cross-flips are indeed necessary?

A *triangulation* G of a closed surface F^2 is a simple graph embedded on the surface such that each face of G is bounded by a 3-cycle and any two faces share at most one edge. By a result of Izvestiev, Klee and Novik [IKN, Theorem 1.1], two different balanced triangulations of a fixed closed surface are connected by a sequence of cross-flips. A cross-flip in dimension d is an operation that exchanges a shellable and co-shellable d -ball in the boundary of the cross $(d+1)$ -polytope with its complement (see [IKN] for the precise definition). In dimension 2, there are 9 different types of cross-flips, but it is known that only 6 flips, described in Figure 1, are necessary (see [IKN, Remark 3.9]). Note that, in Figure 1, it is not allowed to

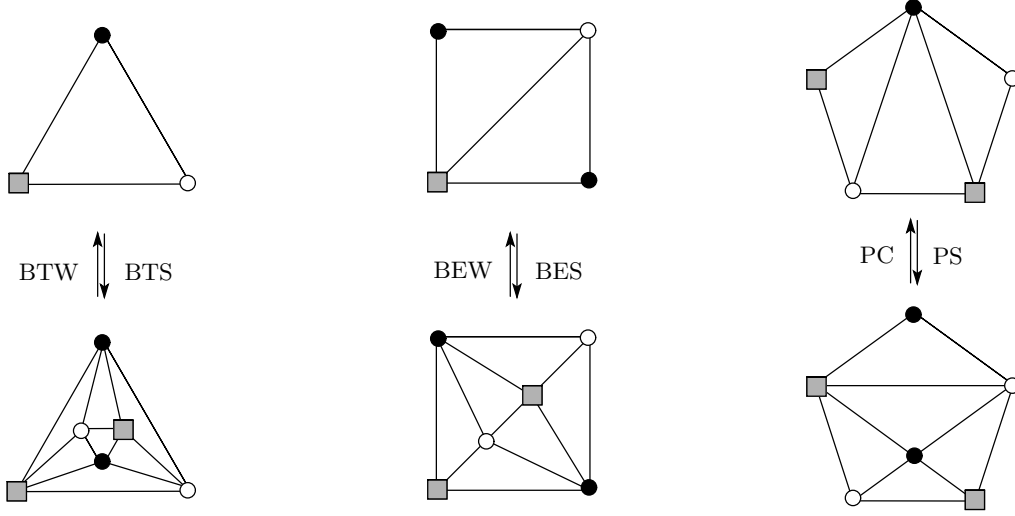


FIGURE 1. Six necessary cross-flips.

make a double edge by the operations and each triangle must be a face. In this paper, we call these six operations, a balanced triangle subdivision (BT-subdivision or BTS), a balanced triangle weld (BT-weld or BTW), a balanced edge subdivision (BE-subdivision or BES), a balanced edge weld (BE-weld or BEW), a pentagon splitting (P-splitting or PS) and a pentagon contraction (P-contraction or PC). A BT-subdivision (resp., -weld) and a BE-subdivision (resp., -weld) are collectively referred to as *balanced subdivisions* (resp., -*welds*). Izmistiev, Klee and Novik [IKN, Problem 3] asked if balanced subdivisions and balanced welds suffice to transform any balanced triangulation of a closed surface into any other balanced triangulation of the same surface. We answer this question.

Theorem 1.1. *For every closed surface F^2 , there are balanced triangulations G and G' of F^2 such that G' cannot be obtained from G by a sequence of balanced subdivisions and welds.*

Next, we consider how many different types of cross flips are necessary. The above result shows that at least a P-contraction or a P-splitting is necessary. Then since we can apply neither a P-contraction nor a P-splitting to the octahedral sphere (the boundary of the cross 3-polytope), we at least need three different cross-flips to transform any balanced triangulation of the 2-sphere to any other balanced triangulation of the 2-sphere. We show that a result proved by Kawarabayashi, Nakamoto and Suzuki in [KNS] implies the following result which guarantees that three flips are indeed enough.

Theorem 1.2. *Any two balanced triangulations of a closed surface F^2 are transformed into each other by a sequence of BE-subdivisions, BE-welds and P-contractions.*

As we mentioned, the set of three moves in the theorem is minimal possible. However, somewhat surprisingly, we show in Theorem 4.3 that most balanced triangulations of a fixed closed surface are actually connected by only P-splittings and P-contractions. In particular, we prove the following strong statement for the 2-sphere.

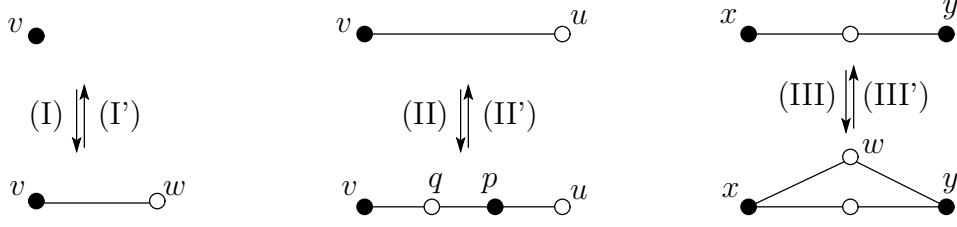


FIGURE 2. Six operations defined for bipartite graphs.

Theorem 1.3. *Any two balanced triangulations of the 2-sphere except the octahedral sphere can be transformed into each other by a sequence of P -splittings and P -contractions.*

This paper is organized as follows. In the next section, we introduce some operations defined for bipartite graphs, and show a key lemma to prove our main theorem. Section 3 is devoted to prove our first main result in the paper. In Section 4, we discuss how many different types of cross-flips are sufficient to connect given two balanced triangulations of a closed surface.

2. OPERATIONS FOR BIPARTITE GRAPHS

In this section, we consider bipartite graphs which are not necessarily embedded on surfaces, and prove the key lemma to prove our first main theorem.

We first introduce some notation. In the paper, we consider simple graphs. Let G be a simple graph. We denote by $V(G)$ the vertex set of G . The *degree* of the vertex v in G is the number of edges of G that contains v . The *minimal degree* of G is the minimum of degrees of vertices of G . An edge on vertices a and b will be denoted by ab and a face on vertices a, b and c will be denoted by abc . A graph G is d -colorable if there is a map $c : V(G) \rightarrow \{1, 2, \dots, d\}$ such that $c(v) \neq c(u)$ for any edge uv of G . A 2-colorable graph is called a *bipartite graph*. For bipartite graphs, we define the following three operations: Let H be a bipartite graph.

- (I) Add a pendant edge vw with $v \in V(H)$ and $w \notin V(H)$. (A pendant edge is an edge such that one of its vertex has degree one.)
- (II) Replace an edge $e = uv$ of H with three edges up, pq, qv , where p and q are new vertices.
- (III) Add a vertex $w \notin V(H)$ and two incident edges xw, wy where x and y have distance 2 in H (i.e., xy is not an edge of H and there is a vertex z such that xz and yz are edges of H).

The inverse operations of the above (I), (II) and (III) are represented by (I'), (II') and (III'), respectively (see Figure 2). In particular, we call (II) the *subdivision* of uv and call (II') the *smoothing* of the edges up, pq, qv . Note that each of these six operations preserves the bipartiteness of the graph.

A set of two adjacent vertices $\{p, q\}$ of degree 2 in a bipartite graph H is said to be *smoothable* if it is possible to apply (II') that removes the vertices p and q to H ; that is, there exists no cycle of length 4 containing p and q . Furthermore, a vertex w of degree 2 in a bipartite graph H is said to be *removable* if we can remove the vertex w by applying (III'); that is, there exists a 4-cycle in H containing w . The

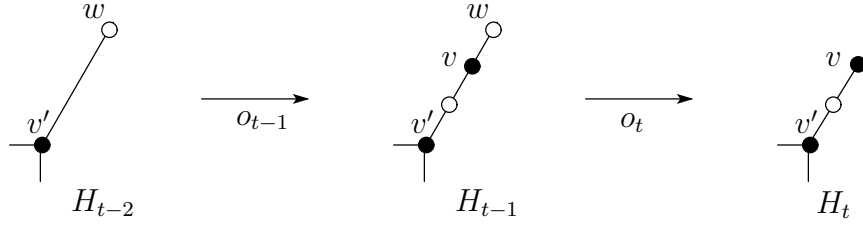


FIGURE 3. Configurations in Case 1 in the proof of Lemma 2.1.

following lemma plays an important role when we prove our main theorem in the next section.

Lemma 2.1. *Let H be a bipartite graph with minimum degree at least 3. If H' is obtained from H by a sequence of operations (I), (II), (III), (I'), (II') and (III'), then H' is obtained from H by a sequence of operations (I), (II) and (III).*

Proof. In the following argument, we say that a bipartite graph is *configurable* from H (by at most t steps) if it can be obtained from H by applying operations (I), (II) and (III) (at most t times).

Let H' be a graph obtained from H by a sequence of operations (I), (II), (III), (I'), (II') and (III'). Then, there is a sequence of bipartite graphs $H = H_0, H_1, \dots, H_t = H'$ such that H_{i+1} is obtained from H_i by one of the six operations for $i = 0, \dots, t-1$, as shown in the following diagram.

$$H = H_0 \xrightarrow{o_1} H_1 \xrightarrow{o_2} H_2 \longrightarrow \dots \xrightarrow{o_{t-2}} H_{t-2} \xrightarrow{o_{t-1}} H_{t-1} \xrightarrow{o_t} H_t = H'.$$

We claim that $H_t = H'$ is configurable from H by at most t steps. We proceed by induction on t . Since any vertex of H has degree at least 3, o_1 must be (I), (II) or (III). Thus the assertion is obvious when $t = 1$. Suppose $t \geq 2$. To prove the desired assertion, it only suffices to show the case when each of o_1, \dots, o_{t-1} is one of (I), (II) and (III), and o_t is one of (I'), (II') and (III').

[Case 1] Suppose that o_t is (I') which removes a vertex w and an edge vw from H_{t-1} . If w is not a vertex of H_{t-2} , then o_{t-1} should be (I) which add w and vw ; note that each of (II) and (III) does not generate a new vertex of degree 1. In this case, it is clear that $H_{t-2} = H_t$ and hence H_t is configurable. Thus, we assume that w is a vertex of H_{t-2} . Since none of (I), (II) and (III) decrease the degrees of vertices, w has degree 1 in H_{t-2} . Let v' denotes the unique neighbor of w in H_{t-2} .

First, suppose that $v \neq v'$. In this case, o_{t-1} should be (II) that subdivide $v'w$, and hence a graph isomorphic to H_t can be obtained from H_{t-2} by adding a pendant edge to w by applying (I) (see Figure 3). Next, we suppose that $v = v'$. We delete a vertex w from H_{t-2} and denote the resulting graph by H'_{t-2} . Since $H_{t-2} \rightarrow H'_{t-2}$ is an operation (I'), by the induction hypothesis, H'_{t-2} is configurable from H by at most $t-1$ steps. Furthermore, since o_{t-1} is not (II) which subdivides vw , we can apply the same operation as o_{t-1} to H'_{t-2} and obtain a graph isomorphic to H_t . Therefore, H_t is configurable from H also in this case.

[Case 2] Suppose that o_t is (II') that replace edges up, pq, qv with uv . First, suppose that both p and q are vertices of H_{t-2} and $\{p, q\}$ is smoothable in H_{t-2} . Let u' and

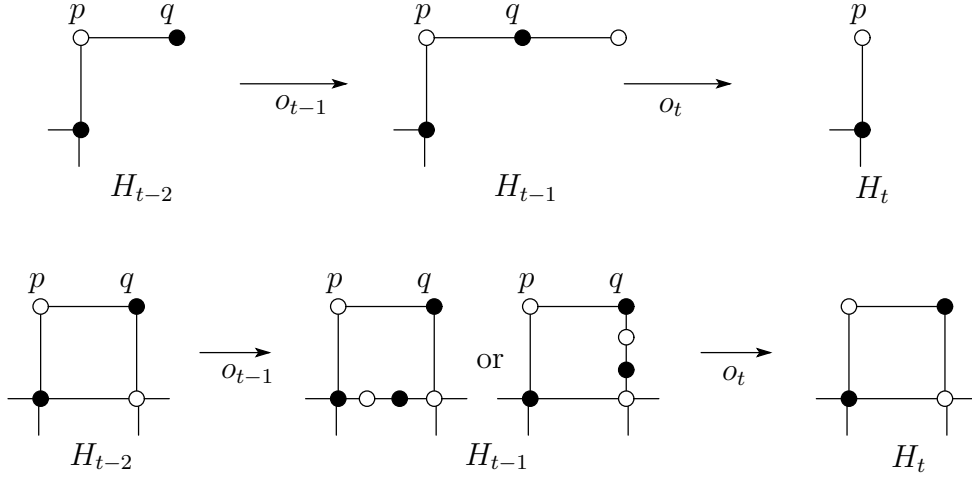


FIGURE 4. Configurations of (C) in Case 2 in the proof of Lemma 2.1.

v' denote the vertices such that $u'p, pq, qv'$ are edges of H_{t-2} . We apply (II') that replace $u'p, pq, qv'$ with $u'v'$ to H_{t-2} and denote the resulting graph by H'_{t-2} . By the induction hypothesis, H'_{t-2} is configurable from H by at most $t-1$ steps. If o_{t-1} is not (II) which subdivides either $u'p$ or qv' , then we can apply o_{t-1} to H'_{t-2} and obtain a graph isomorphic to H_t . On the other hand, if o_{t-1} is (II) that subdivides either $u'p$ or qv' , then H_t and H_{t-2} are clearly isomorphic. In either case, H_t is configurable from H by at most t steps.

By the above argument, we only need to discuss the case when at least one of p and q is not a vertex of H_{t-2} or $\{p, q\}$ is not smoothable in H_{t-2} . We divide the argument into three cases (A), (B) and (C) depending on the situation.

(A) Neither p nor q is a vertex of H_{t-2} : In this case, o_{t-1} is clearly an operation adding p and q , that is, o_{t-1} is (II) that subdivides uv in H_{t-2} . It is easy to see that $H_{t-2} = H_t$.

(B) p is a vertex of H_{t-2} but q is not of H_{t-2} : Note that there exists no cycle of length 4 containing p and q in H_{t-1} since $\{p, q\}$ is smoothable in H_{t-1} . Under the condition, q must be added by o_{t-1} , and we can conclude that o_{t-1} is (II) that subdivide an edge incident to p . (If o_{t-1} is (III), then p and q would lie on a 4-cycle in H_{t-1} .) As a result, H_{t-2} is isomorphic to H_t and hence H_t is configurable from H .

(C) Both of p and q are the vertices of H_{t-2} : Here note that p and q are adjacent and have degree at most 2 in H_{t-2} since each of (I), (II) and (III) does not decrease the degrees of vertices and does not join two non-adjacent vertices. If one of p and q , say q , has degree 1, then o_{t-1} should be (I) that add an edge incident to q since (III) would generate a 4-cycle containing p and q . In this case, a graph isomorphic to H_t can be obtained from H_{t-2} by deleting q using operation (I'), and hence H_t is configurable from H by the induction hypothesis (see the upper diagram in Figure 4). On the other hand, if each of p and q has degree 2, then there should exist a 4-cycle containing p and q in H_{t-2} under our assumption. Since $\{p, q\}$ is smoothable in H_{t-1} , o_{t-1} should be (II) that subdivide an edge on the 4-cycle. In any case, H_{t-2} and H_t is isomorphic to each other (see the bottom diagram in Figure 4).

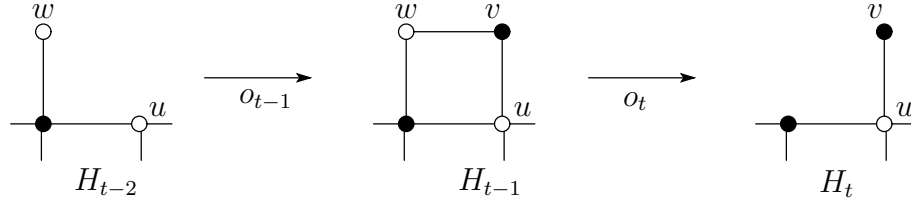


FIGURE 5. Configurations of (B) in Case 3 in the proof of Lemma 2.1.

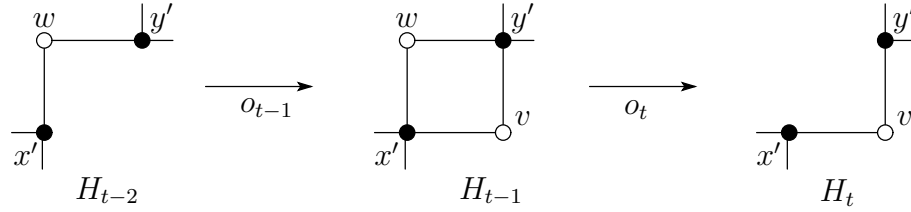


FIGURE 6. Configurations of (C) in Case 3 in the proof of Lemma 2.1.

[Case 3] Suppose that o_t is (III') deleting a vertex w of degree 2 and two edges xw and yw . Note that H_{t-1} must have a 4-cycle that contains w . First assume that w is a vertex of H_{t-2} and is removable in H_{t-2} . Let x' and y' denote the vertices adjacent to w in H_{t-2} . Now, since there exists a 4-cycle containing w in H_{t-1} , o_{t-1} is not (II) that subdivides $x'w$ or wy' . Thus, we have $\{x', y'\} = \{x, y\}$. We delete w from H_{t-2} by applying (III') and denote the resulting graph by H'_{t-2} . Since o_{t-1} is not (II) subdividing xw or wy , we can apply the same operation as o_{t-1} to H'_{t-2} and obtain a graph isomorphic to H_t . By the induction hypothesis, H_t is configurable from H by at most t steps.

By the above argument, we may assume that w is not a vertex of H_{t-2} or w is not removable in H_{t-2} . It suffices to discuss the following three cases (A), (B) and (C).

(A) w is not a vertex of H_{t-2} : Clearly, w must be added by o_{t-1} . Since H_{t-1} has a 4-cycle containing w , o_{t-1} cannot be (II); that is, o_{t-1} should be (III). Then, it is easy to see that $H_{t-2} = H_t$.

(B) w has degree 1 in H_{t-2} : In this case, o_{t-1} is clearly (III). We assume that o_{t-1} adds a vertex v and edges wv and vu (see Figure 5). We remove w from H_{t-2} by applying (I'), and denote the resulting graph by H'_{t-2} . By the induction hypothesis, H'_{t-2} is configurable from H by at most $t - 1$ steps. Furthermore, H_t is obtained from H'_{t-2} by (I) which adds an edge incident to u . Thus, H_t is also configurable from H by at most t steps.

(C) w is a vertex of degree 2 in H_{t-2} : Denote two vertices adjacent to w in H_{t-2} by x' and y' . By our assumption, w is not removable in H_{t-2} , that is, there exists no cycle of length 4 containing w . On the other hand, w is removable and there exists such a 4-cycle in H_{t-1} . To satisfy these conditions, o_{t-1} should be (III) which adds a vertex v and two edges $x'v$ and vy' . However, it is easy to see that H_{t-2} is isomorphic to H_t (see Figure 6).

Now, we have considered all cases and hence the lemma follows. \square

3. PROOF OF THEOREM 1.1

An even embedding H of a closed surface F^2 is a graph embedded on F^2 such that each face of H is bounded by a cycle of even length. For an even embedding H of F^2 , its *face subdivision*, denoted by $S(H)$, is the triangulation of F^2 obtained from H by adding a new vertex into each face of H and joining it all vertices on the corresponding boundary cycle. Since H is 2-colorable and since no vertices of $S(H)$ which are not the vertices of H are adjacent, $S(H)$ is a balanced triangulation. Conversely, for any balanced triangulation G of F^2 , we can obtain an even embedding H of F^2 such that $G = S(H)$ by removing vertices of one color from G . We denote by $e(G)$ the number of edges of a graph G . Since $|V(S(H))|$ equals the sum of $|V(H)|$ and the number of faces of H , by Euler's formula, for even embeddings K and K' of a fixed closed surface F^2 one has $|V(S(K))| > |V(S(K'))|$ if and only if $e(K) > e(K')$.

For each closed surface F^2 , there are infinitely many even embeddings whose minimal degree is at least 3. Hence the next result proves Theorem 1.1.

Theorem 3.1. *Let H and K be even embeddings of a closed surface F^2 whose minimal degree is at least 3. If $S(H)$ is not isomorphic to $S(K)$, then $S(K)$ cannot be obtained from $S(H)$ by a sequence of balanced subdivisions and welds.*

Proof. We may assume $e(H) \geq e(K)$, and in particular $|V(S(H))| \geq |V(S(K))|$. Let $G = S(H)$ and let $G' \neq G$ be a balanced triangulation of F^2 which can be obtained from G by a sequence of balanced subdivisions and welds. To prove the desired statement, it is enough to prove that $|V(G)| < |V(G')|$.

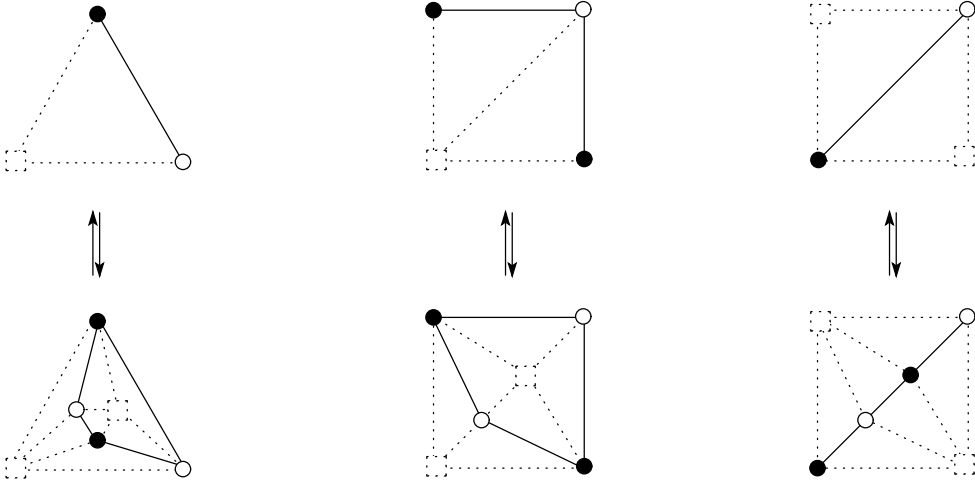


FIGURE 7. Corresponding operations in H .

Since balanced subdivision and welds preserve the balancedness, there is an even embedding H' of F^2 such that $G' = S(H')$ and is obtained from H by a sequence of operations shown in Figure 7 which comes from balanced subdivisions and welds. Furthermore, it is not difficult to check that each operation in Figure 7 can be realized by a combination of the operations (I), (II), (III), (I'), (II') and (III'). Then, by Lemma 2.1, the bipartite graph H' is obtained from H by a sequence of

operations (I), (II) and (III). Recall $G = S(H)$ and $G' = S(H')$. Since $H = H'$ implies $S(H) = S(H')$ and since we assume $G \neq G'$, we have $e(H) < e(H')$, which proves $|V(G)| < |V(G')|$ as desired. \square

Remark 3.2. Face subdivisions $S(H)$ and $S(K)$ could be isomorphic even if $H \neq K$. Indeed, for a balanced triangulation G , one could obtain 3 different even embeddings whose face subdivision is G by removing the vertices of one color from G . On the other hand, it is easy to make even embeddings H and K with $S(H) \neq S(K)$. For example, if $e(H) \neq e(K)$, then we have $|V(S(H))| \neq |V(S(K))|$, and therefore $S(H) \neq S(K)$.

Remark 3.3. The proof of Theorem 3.1 says that, in the theorem, if we assume $e(H) \geq e(K)$, then we do not need to assume that K has minimal degree ≥ 3 . For example, if $S(H)$ is the face subdivision of the cube and $S(K)$ is the octahedral sphere, then $S(K)$ cannot be obtained from $S(H)$ by a sequence of balanced subdivisions and welds.

4. NECESSARY OPERATIONS FOR BALANCED TRIANGULATIONS

In this section, we discuss how many different types of cross-flips are necessary. We first introduce operations called an N -flip and a P_2 -flip originally defined in [NSS], as shown in Figure 8. (An N -flip is also found in cross-flips in [IKN, Figure 1].) Note that it is not allowed to make a double edge by the operations and each triangle in Figure 8 must be a face. Using those operations, Kawarabayashi et al. [KNS] proved the following theorem.

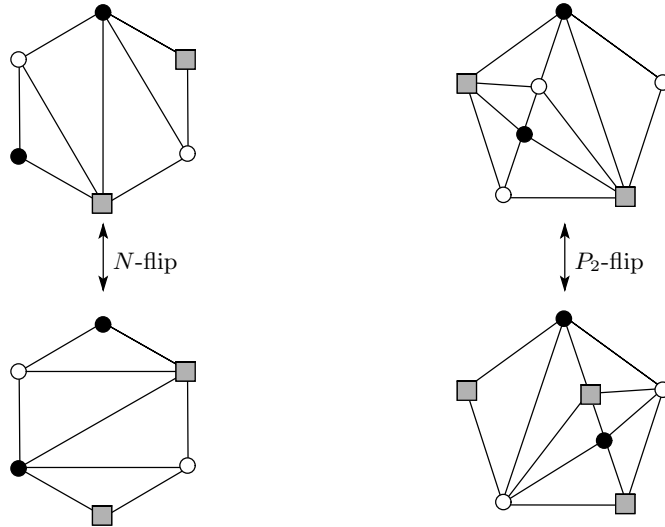


FIGURE 8. N -flip and P_2 -flip.

Theorem 4.1 (Kawarabayashi, Nakamoto and Suzuki [KNS]). *For any closed surface F^2 , there exists an integer M such that any two balanced triangulations G and G' on F^2 with $|V(G)| = |V(G')| \geq M$ can be transformed into each other by a sequence of N - and P_2 -flips.*

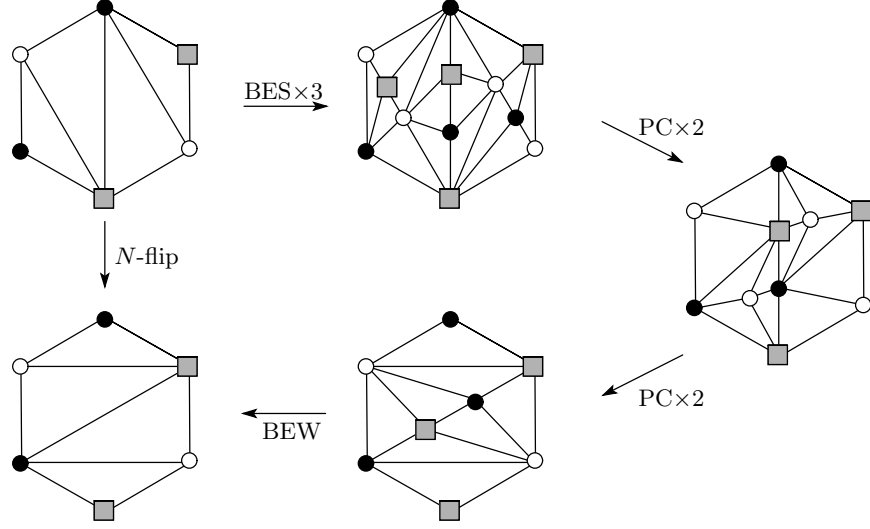
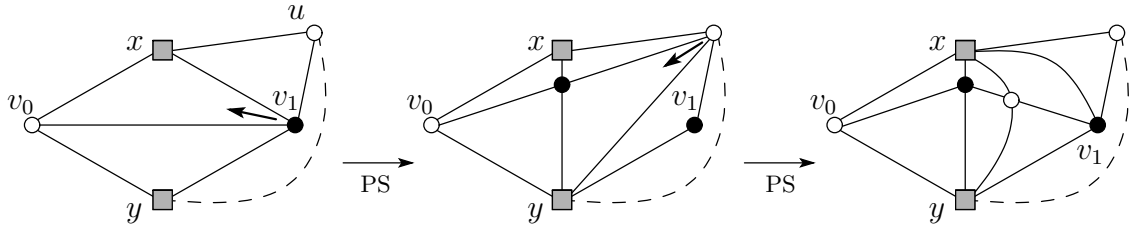
FIGURE 9. An N -flip realized by a sequence of other operations.

FIGURE 10. Two PSs corresponding to a BES.

We now prove Theorem 1.2 in the introduction, saying that BE-subdivisions, BE-welds and P-contractions are enough.

Proof of Theorem 1.2. Clearly, a P_2 -flip can be replaced with a combination of a BE-subdivision and a BE-weld. Furthermore, an N -flip is replaced with a sequence of BE-subdivisions, P-contractions and a single BE-weld, as shown in Figure 9. Since a BE-subdivision increases the number of the vertices by two and a P-contraction decreases the number of the vertices by one, the desired assertion follows from Theorem 4.1. \square

Next, we show that most balanced triangulations of a fixed closed surface F^2 are connected by a sequence of P-contractions and P-splittings. The following simple fact can be observed from Figure 10.

Lemma 4.2. *Let G and G' be balanced triangulations of a closed surface F^2 such that G' is obtained from G by applying the BE-subdivision to the edge v_0v_1 in G . Let xv_0v_1 and yv_0v_1 be the faces of G that contains v_0v_1 and let $u \neq v_0$ be the vertex such that xv_1u is a face of G . If uy is not an edge of G , then G' is obtained from G by a sequence of P-splittings.*

Theorem 4.3. *Any two balanced triangulations of a closed surface F^2 other than finite exceptions (depending on F^2) can be transformed into each other by a sequence of P -splittings and P -contractions.*

Proof. First, observe that each of BE-subdivisions and BE-welds applied in Figure 9 satisfies the assumption of Lemma 4.2. Hence any N -flip can be replaced by a sequence of P -splittings and P -contractions. Similarly, a P_2 -flip is replaced with a combination of P -splittings and P -contractions by Lemma 4.2. This observation also implies that if we can apply either an N -flip or a P_2 -flip to a balanced triangulation, then we can apply a P -splitting.

Now, let G and G' be balanced triangulations of F^2 with $|V(G')| \geq |V(G)| \geq M$ where M is the integer obtained in Theorem 4.1. By Theorem 4.1, G can be transformed into another balanced triangulation with the same number of vertices by a sequence of N - and P_2 -flips. Note that this implies that we can apply a P -splitting to G . After applying a P -splitting to G , we obtain a balanced triangulation of F^2 with $|V(G)| + 1$ vertices. We can repeat the argument until the number of vertices becomes $|V(G')|$; denote the resulting graph by G_0 . By Theorem 4.1 and the above argument G_0 and G' can be transformed into each other by a sequence of P -splittings and P -contractions. Therefore, we conclude that G and G' are connected by only P -splittings and P -contractions. Then the assertion follows since there exist only finitely many balanced triangulations of F^2 with the number of vertices less than M . \square

It would be natural to ask what are the exceptions in Theorem 4.3. Let F^2 be a closed surface and let M be an integer given in Theorem 4.1. The proof of the Theorem 4.3 says that two balanced triangulations are connected by a sequence of P -splittings and P -contractions if they have at least M vertices. We say that a balanced triangulation G of F^2 is *exceptional* if G cannot be connected to a balanced triangulation G' of F^2 with $|V(G')| \geq M$ by a sequence of P -splittings and P -contractions (this condition does not depend on a choice of M). If we can apply a P -splitting to G , that is, there is a graph G' such that G' is obtained from G by a P -splitting, then we can again apply a P -splitting to G' . Thus if we can apply a P -splitting to G , then G is not exceptional. Also, if it is possible to apply a P -contraction to G , then it is also possible to apply a P -splitting to G . Thus we have the following criterion.

Proposition 4.4. *A balanced triangulation G is not exceptional if and only if G have faces vwx, vxy, vyz such that wz is not an edge of G .*

We think that exceptional balanced triangulations are quite rare. Indeed, for the 2-sphere we have the following result, which proves Theorem 1.3.

Theorem 4.5. *The octahedral sphere is the only exceptional balanced triangulation of the 2-sphere.*

Proof. Let G be an exceptional balanced triangulation of the 2-sphere. Since the octahedral sphere is the only triangulation of the 2-sphere all whose vertices have degree 4, it suffices to show that every vertex of G has degree 4.

Let v be a vertex of G and uv an edge of G . We claim that v has degree 4. Let uvx and uvy be the faces of G that contains uv . Also, let $z \neq u$ and $w \neq u$ be the

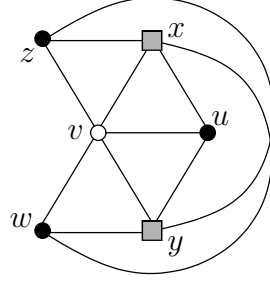


FIGURE 11. A configuration in the proof of Theorem 4.5.

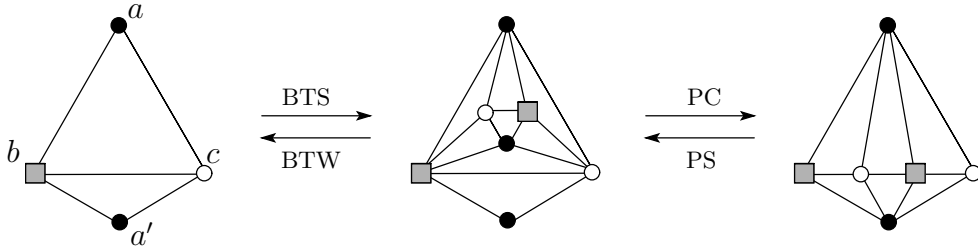


FIGURE 12. Replacement of a BEW and a BES with the other four operations.

vertices such that vxz and $vw y$ are faces of G . Note that $z \neq y$ since they have different colors, and similarly $w \neq x$. By applying Lemma 4.2 to faces vxz, uvx and uvy , we have that yz must be an edge of G . Similarly, by applying Lemma 4.2 to faces $vw y, uv y, uvx$, we have that xw must be an edge of G . Then, since G does not contain the complete bipartite graph of size 3 by the planarity, u must be equal to w , which implies that v has degree 4 as desired (see Figure 11). \square

We close the paper with a few remarks and one question.

Remark 4.6. In Theorem 4.1, it is also true that there is a sequence of N -flips and P_2 -flips that transform G into G' and a given coloring of G into a given coloring of G' (this can be seen from the first paragraph of the proof of [KNS, Theorem 3]). Thus, like [IKN, Theorem 1.1], this stronger property is also true in Theorems 1.2 and 4.3.

Remark 4.7. There is a balanced triangulation of the torus whose underlying graph is the complete tripartite graph $K_{3,3,3}$. By Proposition 4.4, this triangulation is exceptional. We do not know other examples of exceptional balanced triangulations.

Remark 4.8. Any two balanced triangulations of a closed surface F^2 can be transformed into each other by a sequence of BT-subdivisions, BT-welds, P-contractions and P-splittings. Indeed, Figure 12 shows that one can replace BE-subdivisions and BE-welds with combinations of BT-subdivisions, BT-welds, P-splittings and P-contractions.

Remark 4.9. It was asked in [IKN, Problem 4] if two even triangulations of the same combinatorial manifold M with the same coloring monodromy are connected by cross-flips. Since Theorem 4.1 also holds for even triangulations having the

same monodromy, the answer to this problem is yes for closed surfaces. Also, Theorems 1.2 and 4.3 hold in this generality.

Question 4.10. Is there a generalization of Theorem 1.2 (or Theorem 4.3) in higher dimension?

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